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A Relativistic Analogue of the Kepler Problem†

H. P. KÜNZLE

Department of Mathematics, University of Alberta, Edmonton, Canada T6G 2G1

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Abstract

The Poincaré invariant system of two point particles with an instantaneous interaction-at-a-distance originally proposed by Fokker is studied in the Hamiltonian formalism. The interaction, which agrees to first order in the coupling constant with the electromagnetic one obtained from the Liénard–Wiechert fields, is described in an advanced-retarded state space. The first particle moves in the advanced field of the second which in turn is subject to the retarded field of the first. The acceleration terms in the Liénard–Wiechert fields are neglected.

In this theory the state space of the system is a twelve-dimensional manifold Σ and the motions are described as integral curves of a vector field that is obtained as the projection of the generator of time translations in space-time. The Poincaré group acts on this manifold Σ in a well-defined way and leaves a symplectic form ω invariant. Thus the set of all possible motions of this system can be studied by the methods of modern symplectic mechanics. In this paper the general method is explained and the set of all bounded motions for two equal rest masses and an attractive force is studied qualitatively and numerically. In the limit (binding energy)/(sum of rest masses) · (speed of light)$^2 \to 0$ all the features of the classical Kepler motion are obtained.

1. *Introduction*

In two previous papers (Künzle, 1974a and b, to be referred to by I and II, respectively) the general differential geometric formalism was discussed that makes it possible to describe the Poincaré invariant multi-particle systems of instantaneous action-at-a-distance theory in a Hamiltonian form. By the latter we mean a description of the motions of the system as integral curves of a vector field $\mathscr{X}$ on a finite dimensional symplectic manifold (Σ, ω). The dimension of Σ is twice the degree of freedom of the system, the 'time flow' generated by $\mathscr{X}$ leaves ω invariant, i.e. $\mathscr{L}_{\mathscr{X}}\omega = 0$, whence (locally) $\mathscr{X} \lrcorner\, \omega = dH$ for some function H on Σ that can be called the Hamiltonian. For an isolated relativistic system ω is moreover required to be invariant under the Poincaré group.

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It was shown in II that such a state space Σ for the two-particle system can be chosen as the twelve-dimensional submanifold†

$$\{x_1^0 + x_2^0 = 0 = (x_2^\alpha - x_1^\alpha)\eta_{\alpha\beta}(x_2^\beta - x_1^\beta), \qquad x_2^0 \geqslant 0, \qquad \eta_{\alpha\beta}v_k^\alpha v_k^\beta = -1 \quad k = 1, 2\}$$

of the sixteen-dimensional evolution manifold $\mathbb{R}^{16} = \{x_k^\alpha, v_k^\alpha, k = 1, 2\}$. It is thus diffeomorphic to $\mathbb{R}^{12} = \{x_k^A, v_k^A\}$ and describes the positions and 3-velocities of the two particles taken not simultaneously with respect to an observer at rest, but for the second particle at a time retarded with respect to the first. Then it turned out that not only can Σ be equipped with a Poincaré invariant symplectic form ω such that the Poisson brackets‡ between all the position coordinates x_k^A vanish for many non-trivial force laws, but the form ω seemed also to be almost uniquely determined for a given interaction force. It was also shown that the electromagnetic interaction as derived from the Liénard–Wiechert potentials and modified in the way of Fokker (1929) fits very naturally into this formalism.

The purpose of this paper is to study this particular interaction in some detail. We make full use of the symplectic structure ω on Σ and its invariance under the Poincaré group which leads by Noether's theorem (for its formulation in the framework of symplectic geometry see Souriau (1970)) to the existence of the ten well-known integrals of motion. They can be used to introduce a center of mass frame and to define unambiguously a six-dimensional state manifold Σ_r for the 'relative motions' of the two particles which is the direct analogue of the non-relativistic Kepler manifold (Souriau, 1974). The same expressions for the integrals of motion can also be derived using the Fokker action principle as was done by Bruhns (1973) (and partially by Staruszkiewicz (1971)), but it is not obvious from their treatment whether a symplectic structure on a state space and thus a Poisson bracket between arbitrary observables can be defined this way. Bruhns does not define a state space but finds some special solutions of the equations of motion in the four-dimensional formulation. This method has some advantages as it is space-time covariant, but it is not very well suited for a systematic analysis of all solutions since many somewhat arbitrary choices must be made during the integration, like those of suitable curve parameters. Our approach makes it easier to compare the results with the non-relativistic two-body problem and also allows, in principle, a global analysis of the motions by the methods of modern symplectic celestial mechanics.

Section 2 is a summary of the results obtained in I and II. The main definitions and the notation are recalled but for all details and also for more references the reader is referred to these two papers. In Section 3 the integrals of motion

† Greek indices refer to cartesian coordinates of four-dimensional Minkowski space $\mathbb{R}^4$, $\alpha = 0, 1, 2, 3$, $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}\,(-1, 1, 1, 1)$. Capital Latin indices denote the space-like components and range from 1 to 3. For the 3 space components x^A of a vector we also write $\mathbf{x}$.

‡ For $f, g\colon \Sigma \to \mathbb{R}$ define $\{f, g\} := X_f(g)$ where $X_f \lrcorner\, \omega = df$, cf. Abraham & Marsden (1967).

are used to define the center of mass frame in the same way as is customary for special relativistic multiparticle systems that do not interact or interact only by collisions (see, e.g., Synge, 1965). These first integrals are used again in Section 4 to introduce suitable coordinates in the relative state space and to reduce the integration of the equations of motion to two quadratures. Sections 5 and 6, finally, contain the numerical results for the general bounded motions in the case of equal rest masses and to some special questions for the case of arbitrary mass ratios, respectively.

2. *State Space, Dynamical System and Symplectic Structure*

In I and II it was shown that a system of two massive point particles in the instantaneous action-at-a-distance theory can be described by a second-order system

$$\frac{dx_k^\alpha}{dt_k} = a_k v_k^\alpha, \qquad \frac{dv_k^\alpha}{dt_k} = a_k \xi_k^\alpha + b_k v_k^\alpha \tag{2.1}$$

where $k = 1, 2$, and (x_k^α, v_k^α) are cartesian coordinates of the tangent bundle TV_k of Minkowski space $V_k = \mathbb{R}^4$, while a_k and b_k are arbitrary functions on $E = T(V_1 \times V_2)$. The 'accelerations' ξ_k^α are arbitrary given functions on E subject to†

$$v_k^\rho \partial_{\dot\rho_k} \xi_k^\alpha = 2\xi_k^\alpha + \alpha_k v_k^\alpha, \qquad v_l^\rho \partial_{\dot\rho_l} \xi_k^\alpha = \beta_{kl} v_k^\alpha \qquad (l \neq k) \tag{2.2}$$

$$(v_l^\rho \partial_{\rho_l} + \xi_l^\rho \partial_{\dot\rho_l}) \xi_k^\alpha = \gamma_{kl} v_k^\alpha \qquad (l \neq k) \tag{2.3}$$

for some arbitrary functions α_k, β_{kl} and γ_{kl} on E. Invariance of the system (2.1) under the Poincaré group, and thus under its infinitesimal generators on E,

$$\mathscr{T}_\alpha = \partial_{\alpha_1} + \partial_{\alpha_2} \tag{2.4}$$

$$\Omega_{\alpha\beta} = -2 \sum_{k=1}^{2} (x_k^\gamma \eta_{\gamma[\alpha} \partial_{\beta_k]} + v_k^\gamma \eta_{\gamma[\alpha} \partial_{\dot\beta_k]}) \tag{2.5}$$

then implies that the functions ξ_k^α are of the form $\xi_k^\alpha = u_\Sigma^\alpha \tilde\xi_k^\Sigma$ (summed over $\Sigma = 1, 2, 3, 4$) where $u_k^\alpha = v_k^\alpha$ $(k = 1, 2)$, $u_3^\alpha = r^\alpha = x_2^\alpha - x_1^\alpha$,

$$u_4^\alpha \equiv w^\alpha := -\epsilon^\alpha{}_{\lambda\mu\nu} v_1^\lambda v_2^\mu r^\nu \tag{2.6}$$

and‡

$$\tilde\xi_k^l = \tau_k^2 \tau_l^{-2} \xi_k^l, \qquad \tilde\xi_k^3 = \tau_k^2 \xi_k^3, \qquad \tilde\xi_k^4 = \tau_k \tau_l^{-1} \xi_k^4 \tag{2.7}$$

Here $\tau_k := (-\eta_{\alpha\beta} v_k^\alpha v_k^\beta)^{1/2}$ and ξ_k^l, ξ_k^3 and ξ_k^4 are now given functions, depending only on the four parameters‡

† $\partial_{\rho_k} := \partial/\partial x_k^\rho$, $\partial_{\dot\rho_k} := \partial/\partial v_k^\rho$.

‡ We let always $k \neq l = 1, 2$ in the following unless otherwise specified. Assume that v_k^α are future pointing timelike vectors are r^α a future pointing timelike or null vector.

$$\lambda := -\eta_{\alpha\beta} v_1^\alpha v_2^\beta, \qquad \rho_k := -\eta_{\alpha\beta} r^\alpha v_l^\beta \qquad \text{and} \qquad \tau := \sqrt{(-\eta_{\alpha\beta} r^\alpha r^\beta)} \tag{2.8}$$

that satisfy a system of differential equations which determine the τ-dependence if the ξ's are given as functions of λ and ρ_k on a surface τ = const. The quantities $\tilde{\xi}_k^k$ are completely arbitrary functions on E.

While this space-time description is convenient for a discussion of the group invariance properties of the system and the study of several other questions it is obviously not suitable for an explicit, numerical analysis of the possible motions. Instead we choose to describe the motions as integral curves of a vector field $\mathscr{X}$ on a manifold of initial data, or Cauchy surface, Σ. The manifold Σ, which can also be considered as the state space of the system, can be chosen quite arbitrarily, but it is convenient to take a specific submanifold of E such that the coordinates have some physical interpretation.

As in II we choose for Σ the surface $\Sigma = \{z^0 = 0 = \tau, \tau_1 = \tau_2 = 1\}$, where $z^\alpha = \frac{1}{2}(x_1^\alpha + x_2^\alpha)$, i.e. the initial data are the spacelike coordinates x_k^A and v_k^A of the two particles such that the second particle is on a future pointing null ray issuing from the first at the time of measurement and that the timelike component of the 4-velocities are

$$v_k^0 = \sqrt{(1 + v_k^2)}, \qquad v_k^2 := \mathbf{v}_k \cdot \mathbf{v}_k \equiv \delta_{AB} v_k^A v_k^B \tag{2.9}$$

In II it was then shown how a vector field $\mathscr{X}$ on Σ that generates the 'time flow' can be obtained as the projection $\mathscr{X} = -\pi_* \mathscr{T}_0$ of the generator $\mathscr{T}_0$ of time translations on E, where $\pi: E \to \Sigma$ is defined such as to map the point $p = (x_k^\alpha, v_k^\alpha)$ onto the unique point $(\mathbf{x}_k, \mathbf{v}_k)$ on Σ that lies on every integral curve of the system (2.1) that passes through p. Explicitly the dynamical system on Σ has the form (cf. II (2.34))

$$\left.\begin{aligned} \dot{\mathbf{x}}_k &\equiv \frac{d\mathbf{x}_k}{dt} = \mathscr{X}(\mathbf{x}_k) = D^{-1}(1 - w_k)(v_k^0)^{-1}\mathbf{v}_k \\ \dot{\mathbf{v}}_k &\equiv \frac{d\mathbf{v}_k}{dt} = \mathscr{X}(\mathbf{v}_k) = D^{-1}(1 - w_k)(v_k^0)^{-1}\boldsymbol{\xi}_k \end{aligned}\right\} \tag{2.10}$$

where

$$w_k := (rv^0)^{-1}\mathbf{r}\,.\,\mathbf{v}_l, \qquad r := \sqrt{(\mathbf{r}\,.\,\mathbf{r})}, \qquad D := 1 - \tfrac{1}{2}w_1 - \tfrac{1}{2}w_2$$

and

$$\boldsymbol{\xi}_k := \xi_k^l(\mathbf{v}_l - \lambda\mathbf{v}_k) + \xi_k^3(\mathbf{r} - \rho_l\mathbf{v}_k) + \xi_k^4\mathbf{w} \tag{2.11}$$

Here the function ξ_k^Σ are the same as in (2.7) but can now be considered as depending only on λ, ρ_1 and ρ_2.

Equations (2.10) are in principle enough to discuss the motions of the relativistic two-particle system as soon as the functions ξ_k^Σ are given. But just as in non-relativistic mechanics it is more instructive to cast them into canonical form and then to exploit the first integrals obtained via Noether's theorem from the Poincaré invariance of the dynamical system on Σ.

Bringing the dynamical system $(\Sigma, \mathscr{X})$ into canonical form means introducing a symplectic form ω on Σ such that

$$\mathscr{L}_{\mathscr{X}}\,\omega = 0 \tag{2.12}$$

whence (locally) $\mathscr{X} \lrcorner\, \omega = dH$ for some function H on Σ, that is called the Hamiltonian. Then in a canonical coordinate system (q^k, p_k) for which

$$\omega = \sum_{k=1}^{6} dq^k \wedge dp_k$$

(which exists locally according to Darboux's theorem) the system (2.10) becomes

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k}$$

Since equation (2.12) clearly admits many solutions ω for a given $\mathscr{X}$ additional criteria must be used to select the right canonical structure. Physically the most important is clearly that ω should be invariant not just under the time translation $\mathscr{X}$ but under the whole Poincaré group. But also this condition is not restrictive enough.

In all of non-relativistic mechanics one now assumes that ω is such that the position coordinates x_k^A can be chosen as the first half (the q^k's) of a canonical coordinate system. In particular this implies that the Poisson brackets (see footnote ‡, p. 396) between all position coordinates vanish. It is the content of the so-called no-interaction theorems that such a choice of a Pioncaré invariant ω is only possible in the case of the trivial interaction if the position coordinates are measured on a spacelike surface of initial data. In II it was shown, however, that if the surface Σ is defined as we have done here then we can require that

$$\{x_k^A, x_l^B\} = 0 \qquad (k, l = 1, 2) \tag{2.13}$$

on all of Σ and still get reasonable non-trivial interactions. In fact, it can be conjectured that for a given $\mathscr{X}$ there exists at most a two-parameter family of Poincaré invariant symplectic forms satisfying this condition, whereby the two parameters can be interpreted as the rest masses of the two particles.

The accelerations ξ_k^α on the two particles obtained from the Liénard–Wiechert fields in which the acceleration term is ignored have the following invariant components on the surface Σ (cf. I, 6.75).

$$m_k \xi_k^l = (-1)^k g \rho_k^{-3} \rho_l, \qquad m_k \xi_k^3 = -(-1)^k g \lambda \rho_k^{-3}, \qquad \xi_k^4 = 0 \tag{2.14}$$

where $g = e_1 e_2$ is the coupling constant, e_k being the charge and m_k the rest mass of the kth particle. Finding an ω on Σ corresponding to these forces and satisfying (2.13) is not easy in general, but can be done readily to first order in g only. Accepting that approximation as an exact model we slightly modify the force law (2.14), but get much simpler expressions for the integrals of motion.

Rather than stating the explicit form of ω on Σ we give $\pi^*\omega$ in four-dimensionally covariant form. This 2-form can be obtained as the exterior derivative of a 1-form θ that is itself Poincaré invariant and is *on* Σ given by the simple expression†

$$\theta = \sum_k \eta_{\alpha\beta} p_k^\alpha \, dx_k^\beta$$

where the p_k are explicitly‡

$$p_k^\alpha = p_k^\Sigma u_\Sigma^\alpha = m_k v_k^\alpha - g\rho_k^{-1} v_l^\alpha + \tfrac{1}{4} g \rho_1^{-2} \rho_2^{-2} (\rho_l^2 - \rho_k^2 + 2\lambda\rho_1\rho_2) r^\alpha \quad (2.15)$$

Note that in the case of no interaction $(g = 0) p_k^\alpha = m_k v_k^\alpha$ is just the 4-momentum of the kth particle. One can continue thinking of it this way as long as it is kept in mind that in general this term also contains part of the 4-momentum of the interaction.

The symplectic form ω on Σ itself then becomes

$$\omega = -d\iota^*\theta = \iota^*\left(\sum_k \eta_{\alpha\beta} \, dx_k^\alpha \wedge dp_k^\beta \right)$$

and the corresponding 'time flow' vector field $\mathscr{X}$ on Σ has the invariant components

$$\left.\begin{aligned} m_k \xi_k^l &= (-1)^k g \rho_k^{-3} \rho_l \Delta^{-1} (1 + g m_l^{-1} \rho_k \rho_l^{-2}), \qquad \xi_k^4 = 0 \\ m_k \xi_k^3 &= -(-1)^k g \rho_k^{-3} \Delta^{-1} [\lambda + g m_l^{-1} \rho_k \rho_l^{-3} (\lambda\rho_l - \rho_k) - g^2 m_k^{-1} m_l^{-1} \rho_l^{-2}] \end{aligned}\right\} \quad (2.16)$$

where $\Delta = 1 - g m_1^{-1} m_2^{-1} \rho_1^{-1} \rho_2^{-1}$. They agree to first order in g with those in (2.14) and lead to the same 4-accelerations of the two particles that Bruhns (1973) derived from a simple Fokker action. The expressions are relatively complicated but fortunately will not be needed for the integration of the equations of motion.

3. *Center of Mass Frame and Space of Relative Motions*

Noether's theorem states that

$$\tilde{f}_A := \mathbf{A} \lrcorner \theta \quad (3.1)$$

is an integral of motion for any solution of equations (2.1) whenever the vector field $\mathbf{A}$ on E satisfies $\mathscr{L}_\mathbf{A}\theta = 0$, i.e. $\tilde{f}_A$ is in fact of the form $\tilde{f}_A = \pi^* f_A$ for

† Off Σ the coefficients of dv_k^α do not vanish, but since $\omega = \iota^*\pi^*\omega = -d(\iota^*\theta)$, where $\iota: \Sigma \to E$ is the imbedding map. θ need only be known on Σ.

‡ Actually the term $p_2^3 - p_1^3$ remains completely arbitrary, but it does not enter either ω nor the integrals of motion.

some function $f_A : \Sigma \to \mathbb{R}$ which can be obtained simply by evaluating (3.1) on Σ. Moreover, for the Poisson brackets of these first integrals one has

$$\{f_A, f_B\} = f_{[AB]} . \tag{3.2}$$

Substituting for **A** the generators of the Poincaré group on E given in (2.4) and (2.5) yields the following first integrals (evaluated on Σ)

$$\begin{aligned} P^\alpha &= \eta^{\alpha\beta} \mathscr{T}_\beta \lrcorner \theta = p_1^\alpha + p_2^\alpha \\ &= (m_1 - g\rho_2^{-1}) v_1^\alpha + (m_2 - g\rho_1^{-1}) v_2^\alpha + g\lambda \rho_1^{-1} \rho_2^{-1} r^\alpha \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} J^{\alpha\beta} &= \eta^{\alpha\mu}\eta^{\beta\nu}\Omega_{\mu\nu} \lrcorner \theta = -2 \sum_k x_k^{[\alpha} p_k^{\beta]} \\ &= -2(m_1 - g\rho_2^{-1}) z^{[\alpha} v_1^{\beta]} - 2(m_2 - g\rho_1^{-1}) z^{[\alpha} v_2^{\beta]} \\ &\quad -2g\lambda\rho_1^{-1}\rho_2^{-1} z^{[\alpha} r^{\beta]} + (m_1 + g\rho_2^{-1}) r^{[\alpha} v_1^{\beta]} - (m_2 + g\rho_1^{-1}) r^{[\alpha} v_2^{\beta]} \end{aligned} \tag{3.4}$$

namely the total 4-momentum and the total 4-angular momentum tensor, respectively. According to (3.2), regarded as functions on Σ, they form the Lie algebra of the Poincaré group with respect to the Poisson brackets:

$$\begin{aligned} &\{P^\alpha, P^\beta\} = 0, \qquad \{P^\alpha, J^{\beta\gamma}\} = 2\eta^{\alpha[\beta} P^{\gamma]} \\ &\{J^{\alpha\beta}, J^{\gamma\delta}\} = 2J^{\alpha[\gamma}\eta^{\delta]\beta} - 2J^{\beta[\gamma}\eta^{\delta]\alpha} \end{aligned} \tag{3.5}$$

These quantities are well known for systems of non-interacting particles (or particles that only interact via collisions, see, e.g., Synge (1965)). We can treat them here in exactly the same way. Define the *polarization vector*,

$$W^\alpha := \tfrac{1}{2}\epsilon^{\alpha\beta\lambda\mu} P_\beta J_{\lambda\mu} = -(m_1 m_2 - g^2 \rho_1^{-1} \rho_2^{-1}) w^\alpha , \tag{3.6}$$

the total *mass-energy* M by

$$M^2 = -P_\alpha P^\alpha \tag{3.7}$$

and the magnitude L of the intrinsic angular momentum (spin) by

$$M^2 L^2 = W^\alpha W_\alpha \tag{3.8}$$

For physical reason we will always assume that $M > 0$ and $L \geqslant 0$ and more particularly that the total 4-momentum is a future pointing timelike vector. Let also

$$L^\alpha = M^{-1} W^\alpha \tag{3.9}$$

be the spin-4-vector (a spacelike vector orthogonal to P^α) and

$$L^{\alpha\beta} := -M^{-1} \epsilon^{\alpha\beta\lambda\mu} P_\lambda L_\mu \tag{3.10}$$

the *4-spin tensor.*† Then $J^{\alpha\beta} - L^{\alpha\beta}$ should be regarded as the *orbital angular momentum tensor* and it is easily seen that it can be written in the form

$$J^{\alpha\beta} - L^{\alpha\beta} = -2X^{[\alpha}P^{\beta]} \tag{3.11}$$

Contracting (3.11) with P_β gives

$$M^2 X^\alpha + (X^\rho P_\rho)P^\alpha = J^{\alpha\rho}P_\rho \, . \tag{3.12}$$

The 4-vector X^α is thus defined up to an arbitrary term parallel to P^α. It can be considered as defining the worldline of the center of mass of the system.

It can be verified that the spacelike coordinates X^A satisfy $\{X^A, P_B\} = \delta^A_B$, but $\{X^A, X^B\} \neq 0$ unless the spin vanishes. Thus the X^A's are not pure position coordinates, which is not surprising, since the center of mass of a relativistic system depends also on the velocities of the particles. Stated in more mathematical terms this also means that there is no barycentric decomposition of the symplectic realization of the Poincaré group on the state space of a multiparticle system (cf. Souriau, 1970).

In spite of this one can introduce a state space for the relative motions of the system, namely a submanifold Σ_r of Σ obtained by fixing the values of $\mathbf{P}$ and $\mathbf{X}$. In contradistinction to the non-relativistic situation the induced symplectic structure on this six-dimensional manifold Σ_r depends then on the value of $\mathbf{P}$ (but not the value of $\mathbf{X}$) though not in a very essential way. (Cf. II and, for a very general treatment of such reductions, Marsden & Weinstein, 1974.)

It does not seem unreasonable to discuss the motions with respect to a center of mass frame, namely to consider the submanifold

$$\Sigma_r = \{\mathbf{X} = 0 = \mathbf{P}\}$$

of Σ. The condition that $\mathbf{P} = 0$, implies explicitly, in view of (3.3) that on Σ_r

$$(m_1 - g\rho_2^{-1})\mathbf{v}_1 + (m_2 - g\rho_1^{-1})\mathbf{v}_2 + g\lambda\rho_1^{-1}\rho_2^{-1}\mathbf{r} = 0 \tag{3.13}$$

Equation (3.12) together with (3.4) and $\mathbf{X} = 0$ gives

$$\begin{aligned} M\mathbf{z} = &-\tfrac{1}{2}(m_1 + g\rho_2^{-1})r\mathbf{v}_1 + \tfrac{1}{2}(m_2 + g\rho_1^{-1})r\mathbf{v}_2 \\ &+ \tfrac{1}{2}(m_1 v_1^0 - m_2 v_2^0 + g v_1^0 \rho_2^{-1} - g v_2^0 \rho_1^{-1})\mathbf{r} \end{aligned} \tag{3.14}$$

As coordinates for Σ_r we now choose $\mathbf{r}$ and

$$\mathbf{v} := 2N^{-1}(\rho_2\mathbf{v}_2 - \rho_1\mathbf{v}_1) \tag{3.15}$$

where $N := v_1^0\rho_1 + v_2^0\rho_2$, because then, according to (2.10)

$$\dot{\mathbf{r}} = \mathscr{X}(\mathbf{r}) = \mathbf{v} \tag{3.16}$$

Contracting (3.13) with $\mathbf{v}_k$ and $\mathbf{r}$ gives

$$Mr = m_1\rho_2 + m_2\rho_1 - 2g \tag{3.17}$$

† This definition is somewhat controversial. For a recent general discussion see Lorente & Roman (1974).

and

$$Mv_k^0 = m_k + m_l\lambda - g\rho_l^{-1} \tag{3.18}$$

Now solving (3.13) and (3.15) for $\mathbf{v}_k$ and using (3.17) and (3.18) yields

$$Mr\mathbf{v}_k = \tfrac{1}{2}(-1)^k N(m_l - g\rho_k^{-1})\mathbf{v} - g\lambda\rho_k^{-1}\mathbf{r}$$

and substitution of this into (3.14) then expresses $\mathbf{z}$ in terms of $\mathbf{r}$ and $\mathbf{v}$,

$$2M^2\mathbf{z} = [m_1^2 - m_2^2 + g^2(\rho_1^{-2} - \rho_2^{-2})]\,\mathbf{r} + (m_1 m_2 - g^2\rho_1^{-1}\rho_2^{-1})N\mathbf{v} \tag{3.19}$$

Since

$$\mathbf{x}_k = \mathbf{z} + \tfrac{1}{2}(-1)^k\mathbf{r} \tag{3.20}$$

we see that once the time development of $\mathbf{r}$ and $\mathbf{v}$ is known the trajectories of both particles in this center of mass system can be easily obtained. Moreover, we see that Σ_r can be parametrized by all values of $\mathbf{r}$ and $\mathbf{v}$, i.e. that $\Sigma_r \cong \mathbb{R}^3 \times \mathbb{R}^3$.

Instead of the submanifold $\iota_r \colon \Sigma_r \to \Sigma$ with its induced symplectic structure $\omega_r = \iota_r^*\omega$ the relative state space could have been obtained more abstractly by applying the reduction technique of Marsden & Weinstein (1974) to (Σ, ω), which admits the whole Poincaré group as a symplectic symmetry group, and to 'divide out' the space translation subgroup. It then follows that the time translations and the three-dimensional rotation group still act by symplectomorphisms on (Σ_r, ω_r). This can also be seen directly by noting that the vector field $\mathscr{X}$ on Σ is tangent to Σ_r and thus induces a vector field $\mathscr{X}_r$ on Σ_r by restriction, the 'time flow' generator on Σ_r.

Incidently, although the mathematical construction of $\mathscr{X}_r$ on Σ_r is rather straightforward and natural and there is no doubt that the parameter t of these integral curves represents 'time', the physical notions of the exact relationship between a description of the motion as worldlines in space-time and as a flow on a state space seem much more vague. Nevertheless it seems reasonable to interpret the 'time' defined by $\mathscr{X}_r$ on Σ_r as the *proper time of an observer stationed at the center of mass.*

Similarly, the generators of space rotations on Σ,

$$\pi_*(\Omega_{AB}) = -2\sum_k (\underset{k}{x}_{[A}\,\partial_{B_k]} + \underset{k}{v}_{[A}\,\partial_{\dot{B}_k]})$$

are tangent to Σ_r. In fact, it is easy to see that the action of $SO(3)$ on Σ_r is just the direct product of the standard action on $\mathbb{R}^3$ with itself. The integrals of motion corresponding to these symmetries can again be obtained simply by restricting the previously calculated ones to Σ_r. Thus we still have the mass-energy function M and the spin-3-vector $\mathbf{L}$ as integrals of motion on Σ_r. (Note that $L^\alpha = (0, \mathbf{L})$ since $P^\alpha L_\alpha = 0$ and $P^\alpha = M\delta^\alpha_0$.) Explicitly from (3.3) and (3.7),

$$M^2 = m_1^2 + m_2^2 + 2m_1m_2\lambda - 2g(m_1\rho_2^{-1} + m_2\rho_1^{-1}) - g^2K\rho_1^{-2}\rho_2^{-2} \tag{3.21}$$

where

$$K := -\rho_1^2 - \rho_2^2 + 2\lambda\rho_1\rho_2 = w^\alpha w_\alpha \tag{3.22}$$

and from (3.4) and (3.9) with the help of (3.17) and (3.18)

$$\mathbf{L} = \tfrac{1}{2}(Mr)^{-1}N(m_1m_2 - g^2\rho_1^{-1}\rho_2^{-1})\mathbf{r}\times\mathbf{v} \tag{3.23}$$

In these equations the quantities ρ_k, λ and v_k^0 must, of course, be considered as functions of $\mathbf{r}$ and $\mathbf{v}$ as defined, rather implicitly, in (3.13) and (3.15).

It would now be nice if one could proceed to classify globally the motions by looking at the submanifolds of Σ_r defined as inverse images of points in $\mathbb{R}^4$ under the map $(M, \mathbf{L})$: $\Sigma_r \to \mathbb{R}^4$ (cf. Smale, 1970). In practice, however, it is almost impossible to calculate explicitly something like the Jacobian of this map, due to the implicit definition of ρ_k, v_k^0 and λ in terms of $\mathbf{r}$ and $\mathbf{v}$. It may therefore be worthwhile to first study some properties of these motions in a more simple-minded approach, in order to establish whether they seem physical enough to warrant a further, possibly global study.

4. *First Integrals and Reduction to Quadratures*

As it is always done with the classical Kepler problem we will from now on assume (tentatively) that $\Sigma_r = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ only, thus eliminating the obvious singularity in the Coulomb force. But we may be restricting Σ_r further by requiring that the 4-vectors v_k^α, r^α and P^α are future pointing, v_k^α timelike and unit, P^α timelike and r^α null.† This implies in particular that (now always on Σ_r)

$$M > 0, \qquad \rho_k > 0, \qquad N = v_1^0\rho_1 + v_2^0\rho_2 > 0$$
$$v_k^0 \geqslant 1, \qquad v_k^0 = 1 \qquad \text{iff } \mathbf{v}_k = 0$$
$$\lambda \geqslant 1, \qquad \lambda = 1 \qquad \text{iff } \mathbf{v}_1 = \mathbf{v}_2$$
$$K = w^\alpha w_\alpha \geqslant 0, \qquad K = 0 \qquad \text{iff } r^\alpha, v_1^\alpha, v_2^\alpha \text{ are coplanar.}$$

Assume from now on that the coupling constant $g \neq 0$. It can then be used to replace all the quantities by dimensionless ones.‡ Define

$$\kappa := |g|^{-1}g = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \text{ for an } \begin{Bmatrix} \text{attractive} \\ \text{repulsive} \end{Bmatrix} \text{ force} \tag{4.1}$$

$$m := m_1 + m_2 \text{ (total rest mass)} \tag{4.2}$$

$$q := m_2 m_1^{-1}, \qquad \nu := m^{-1}(m_2 - m_1) = (q-1)/(q+1), \qquad \mu := 1 - \nu^2 \tag{4.3}$$

then $m_k = \frac{1}{2}m[1 + (-1)^k\nu]$ and the Newtonian *reduced mass* becomes $m_1m_2m^{-1} = \frac{1}{4}\mu m$. Now let

$$\boldsymbol{\rho} := |g|^{-1}m\mathbf{r}, \qquad \rho := (\boldsymbol{\rho}.\boldsymbol{\rho})^{1/2} \tag{4.4}$$

† Actually it might be physically reasonable, though not compelling, to require that p_k^α are future pointing timelike vectors.

‡ Note that so far we used relativistic units (speed of light = 1). Thus all velocities are already dimensionless.

$$\boldsymbol{\zeta} := |g|^{-1} m\mathbf{z}, \qquad \mathbf{y}_k := |g|^{-1} m\mathbf{x}_k \tag{4.5}$$

$$\tau :- |g|^{-1} mt \tag{4.6}$$

$$\pi := \tfrac{1}{2}|g|^{-1} m(\rho_1 + \rho_2), \qquad \sigma = \tfrac{1}{2}|g|^{-1} m(\rho_1 - \rho_2) \tag{4.7}$$

whence by (3.22)

$$k := |g|^{-2} m^2 K = 2(\lambda - 1)\pi^2 - 2(\lambda + 1)\sigma^2 \qquad (\geqslant 0) \tag{4.8}$$

Note that $\pi > |\sigma|$ on all of Σ_r. For the integrals of motion mass-energy M and spin $\mathbf{L}$ we introduce

$$E := \epsilon(M - m) \tag{4.9}$$

where $\epsilon = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ and $E (\geqslant 0)$ is interpreted as the $\begin{Bmatrix} \text{internal} \\ \text{binding} \end{Bmatrix}$ energy of the system, and let

$$e := E/m \qquad (\geqslant 0) \tag{4.10}$$

If $\epsilon = -1$, which will turn out to be the case of bounded motions for an attractive force, then $0 \leqslant e < 1$.† Define the dimensionless spin 3-vector by

$$\mathbf{l} := |g|^{-1}\mathbf{L} \tag{4.11}$$

and let $l = (\mathbf{l}\,.\,\mathbf{l})^{1/2}$ be its magnitude. The following abbreviations will be convenient:

$$c := 1 + \epsilon e \qquad (>0), \qquad s := \sqrt{[\epsilon(c^2 - 1)]} \qquad (\geqslant 0) \tag{4.12}$$

$$\alpha := 2c(\pi^2 - \sigma^2) \qquad (>0) \tag{4.13}$$

and

$$\beta := \mu\alpha - 8c \tag{4.14}$$

Now (3.17) and (3.18) translate into

$$\pi = c\rho - \nu\sigma + 2\kappa \tag{4.15}$$

and

$$v_k^0 = \tfrac{1}{2}c^{-1}\,[\lambda + 1 - (-1)^k \nu(\lambda - 1)] \tag{4.16}$$

respectively. Solving (4.8) for λ, and substitution into (4.16) gives

$$v_k^0 = 2\alpha^{-1}\,[\pi^2 - \kappa\pi + \tfrac{1}{4}k + (-1)^k(\kappa\sigma - \nu\sigma^2 - \tfrac{1}{4}\nu k)] \tag{4.17}$$

and

$$N = 4|g|m^{-1}\alpha^{-1}\delta \tag{4.18}$$

† A rough idea of the magnitudes involved may be got from these estimates. If two masses of $10^a g$ each circle around each other at 10^b cm distance under gravitational attraction then $e \approx 2\,.\,10^{a-b-29}$. For the ground level of the H-atom or positronium $e \approx 7\,.\,10^{-6}$.

where

$$\delta := \pi^3 + \nu\sigma^3 - \kappa(\pi^2 + \sigma^2) + \tfrac{1}{4}k(\pi + \nu\sigma) \qquad (>0) \tag{4.19}$$

The 'energy integral' (3.21) now becomes

$$2c\alpha[\alpha - 2\rho(\pi - \nu\sigma)] = k\beta \tag{4.20}$$

Conservation of the magnitude of the spin vector is expressed on the one hand by

$$L^2 = L^\alpha L_\alpha = (m_1 m_2 - g^2\rho_1^{-1}\rho_2^{-1})w^\alpha w_\alpha$$

or in dimensionless form by

$$4cl\alpha = k^{1/2}|\beta| \tag{4.21}$$

On the other hand, taking the square of (3.23) and introducing the angle γ between $\boldsymbol{\rho}$ and $\mathbf{v}$,

$$2ML = Nv|m_1 m_2 - g^2\rho_1^{-1}\rho_2^{-1}||\sin\gamma|$$

or

$$2cl\alpha^2 = v\delta\,|\beta\sin\gamma| \tag{4.22}$$

We need two more equations to determine the relations between the different sets of coordinates. Recalling (3.16) or

$$\dot{\boldsymbol{\rho}} = \mathbf{v} \tag{4.23}$$

where from now on $\cdot = d/d\tau$ we have

$$\dot{\rho} = v\cos\gamma\,. \tag{4.24}$$

Comparing this with

$$dr/dt = 2N^{-1}(v_2^0\rho_2 - v_1^0\rho_1)$$

which is obtained directly from (2.10) (or see II (2.41)) one deduces

$$\delta v\cos\gamma + \tfrac{1}{2}k(\nu\pi + \sigma) + 2c\rho\pi\sigma = 0 \tag{4.25}$$

It will turn out that equations (4.20) to (4.25) suffice to determine the dependence on τ of $\boldsymbol{\rho}$ and $\mathbf{v}$. Once this is known, however, we can compute $\boldsymbol{\zeta}$ and $\mathbf{y}_k$ according to (3.14) and (3.20), which now become

$$\boldsymbol{\zeta} = -\tfrac{1}{2}c^{-2}\alpha^{-2}\,[(16c^2\pi\sigma + \nu\alpha^2)\boldsymbol{\rho} - \beta\,\delta\mathbf{v}] \tag{4.26}$$

$$\mathbf{y}_k = \boldsymbol{\zeta} + \tfrac{1}{2}(-1)^k\boldsymbol{\rho} \tag{4.27}$$

respectively.

Zero Angular Momentum

If the angular momentum vanishes one should expect to get only straight line motions. Unfortunately this does not yet follow mathematically from equations (4.20) to (4.25). Rather it appears that a further restriction of the manifold Σ_r is needed to get only the physically meaningful solutions.

If $l = 0$ equation (4.21) implies $k = 0$ and/or $\beta = 0$.

Case A. Assume $\beta \neq 0$, hence $k = 0$. Then (4.22) gives $v \sin \gamma = 0$. Thus $\boldsymbol{\rho}$ and $\mathbf{v}$ are always parallel which corresponds to *straight line motion* (relatively, as well as for the two trajectories in the center of mass system according to (4.26) and (4.27)). Then (4.20) and (4.25) become

$$\alpha = 2\rho(\pi - \nu\sigma) \tag{4.28}$$

and

$$\dot{\rho} = -2c\rho\pi\sigma\delta^{-1} \tag{4.29}$$

respectively. Equations (4.28), (4.13) and (4.15) yield

$$4c + 2\kappa(1 + 2\epsilon s^2)\rho + \epsilon s^2 c\rho^2 - 2\epsilon\nu s^2\rho\sigma - 4\kappa\nu c\sigma - \mu c\sigma^2 = 0 \tag{4.30}$$

Solving this equation for σ as a function of ρ and substituting into (4.29) then gives an equation of the form $d\tau = f(\rho)\, d\rho$ which determines the time development of the straight line motion completely. To find the range of ρ in which the motion will take place one must satisfy the condition $\pi > |\sigma|$. The turning points are obtained for $\sigma = 0$ (according to (4.29)). For $\nu = 0$ (equal rest masses) this is easily done explicitly with the results:

	attractive force	repulsive force
$M < m$	$0 < \rho \leqslant 2cs^{-2}$	no motion
$M = m$	$0 < \rho$	no motion
$M > m$	$0 < \rho$	$2cs^{-2} \leqslant \rho$

where the relative velocity vanishes precisely when $\rho = 2cs^{-2}$. Thus this case corresponds qualitatively in every respect to the classical Kepler problem.

Case B. If we let instead $\beta \equiv 0$ in (4.21), then k need not necessarily vanish. Again assuming $\nu = 0$ for simplicity one finds using equations (4.20) to (4.25) and checking the condition $|\sigma| < \pi$, $k \geqslant 0$ that there is just one admissible case, namely for a repulsive force with $\epsilon = 1$ and $e = \sqrt{(2)} - 1$. Then $\rho \equiv 2\sqrt{2}$, $\sigma = 0$ thus also $\dot{\rho} \equiv 0$, but k ($\geqslant 0$) is arbitrary. This case therefore represents a circular (relative) motion with vanishing angular momentum! We will see later in the case $l \neq 0$ that it may be necessary to limit the physical relative state space to a domain where $\beta > 0$.

Non-zero Angular Momentum

If $\mathbf{l} \neq 0$ choose an orthonormal basis such that $\mathbf{l} = l\mathbf{e}_3$. Then by (3.23) $\boldsymbol{\rho}$ and $\mathbf{v}$ lie both in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ and are not parallel. The motion will therefore be confined to this plane. Let

$$\boldsymbol{\rho} = \rho(\cos\phi, \sin\phi)$$

and

$$\mathbf{v} = v(\cos\psi, \sin\psi)$$

where $\gamma = \phi - \psi \neq 0, \pi$.

We have again (4.24) and also

$$\dot{\phi} = -v\rho^{-1}\sin\gamma \tag{4.31}$$

For definiteness we can assume that $\sin\gamma < 0$ such that $\dot{\phi} > 0$. Now (4.21) implies $\beta \neq 0 \neq k$ and

$$k = 16c^2 l^2 \alpha\beta^{-2}$$

Equations (4.20), (4.22) and (4.25) then become

$$\Phi := \beta[\alpha - 2\rho(\pi - \nu\sigma)] - 8cl^2\alpha = 0 \tag{4.32}$$

$$\dot{\phi} = 2cl\alpha^2\rho^{-1}\delta^{-1}|\beta|^{-1} \tag{4.33}$$

and

$$\dot{\rho} = -2c\delta^{-1}[\rho\pi\sigma + 4cl^2\alpha^2\beta^{-2}(\nu\pi + \sigma)] \tag{4.34}$$

respectively. These equations effectively reduce the problem to quadratures. In principle the motions are found as follows:

Step 1. Solve $\Phi = 0$ for $\sigma = \Sigma(\rho; e, \lambda, \nu)$, whence $\pi = \Pi(\rho; e, \lambda, \nu)$.

Step 2. For each solution Σ check that (i) Σ real, (ii) $\pi > 0$, (iii) $\alpha > 0$, (iv) $\beta \neq 0$, (v) $\delta > 0$, if not discard this solution in the particular parameter range.

Step 3. Substitute Σ into (4.33) and (4.34) and integrate over ρ.

5. *Numerical Results for Equal Rest Masses*

We consider from now on only the case of an attractive force ($\kappa = 1$) and bounded motion ($\epsilon = -1$, i.e. $M < m$) for non-zero angular momentum ($l > 0$). It turns out that the solution of (4.32) for σ in terms of ρ is possible in closed form only if $\nu = 0$, that is, for two particles with equal rest masses. The problem is reduced to two integrations which are relatively easy to carry out by computer, especially in the case of bounded motion (i.e. $0 < \rho_1 \leqslant \rho \leqslant \rho_2$).

Equation (4.32) for $\nu = 0$ is equivalent to

$$\alpha^2 - 2\alpha[\rho\pi + 4c(1 + l^2)] + 16c\rho\pi = 0 \tag{5.1}$$

where now $\pi = c\rho + 2$. Solving (5.1) for α gives

$$\alpha = 8c + K + \epsilon_1\Delta = \beta + 8c \tag{5.2}$$

where $K := \rho\pi + 4c(l^2 - 1)$ and $\Delta := \sqrt{(K^2 + 64c^2 l^2)}$. Since $\Delta > |K|$ it follows from (5.2) that $\epsilon_1 = \mathrm{sgn}(\beta)$. Now (4.13) yields

$$\sigma = -\epsilon_2(2c)^{-1/2} I^{1/2}, \qquad I := 2c\pi^2 - \alpha$$

where $\epsilon_2 = \pm 1$. Since σ must be real the motion is confined to values of ρ for which $I \geqslant 0$. In this range we then have

$$d\phi/d\rho = \epsilon_1 \epsilon_2 H I^{-1/2} \tag{5.3}$$

and

$$d\tau/d\rho = \epsilon_2 G I^{-1/2} \tag{5.4}$$

where

$$G := (2c)^{-3/2} [\beta^2(\alpha + 2c^2 \rho\pi^2) + 8c^3 l^2 \alpha^2 \pi] (\beta^2 \rho\pi + 4\pi^2 c\alpha^2)^{-1}$$

and

$$H := (2c)^{1/2} l\alpha^2 |\beta| \rho^{-1} (\beta^2 \rho\pi + 4l^2 c\alpha^2)^{-1}$$

The sign ϵ_2 is chosen positive for outward and negative for inward motion, but ϵ_1 is a bit harder to decide on. Since $\beta \neq 0$ along every motion with $l \neq 0$ and since the physically most reasonable states turn out to have $\beta > 0$ we may consider this as a restriction of the state manifold. Moreover, only if this condition is adopted $\dot{\rho}$ and $\dot{\phi}$ will be continuously differentiable. We will henceforth assume it. Then $I \geqslant 0$ is equivalent to

$$F(\rho) := s^2 c\rho^3 - 2(1 - 3s^2)\rho^2 + 4c(l^2 - 2)\rho + 8l^2 \leqslant 0 \tag{5.5}$$

For any value of e in $(0, 1)$ it is easily seen that $F(\rho) = 0$ has either no positive roots, two coinciding ones or two distinct positive roots depending on the value of l. These cases correspond to no motion, a stable circular motion (ρ = const.) and a bounded motion between a perihelion distance ρ_1 and an aphelion distance ρ_2.

Circular Motion

For a given energy e a circular motion results if $l = l_c$, the maximum value for which $F = 0$ has a positive root. Thus $l_c(e)$ and $\rho_c(e)$, the radius of the circular motion, are obtained as the solutions of the system

$$F(\rho, l; e) = 0 \qquad \text{and} \qquad \partial F/\partial\rho\,(\rho, l; e) = 0 \tag{5.6}$$

namely

$$\rho_c = \tfrac{1}{3} s^{-2} c^{-1} (1 - 6s^2 + u_1 + u_2) \tag{5.7}$$

$$l_c^2 = 2 + (1 - 3s^2) c^{-1} \rho_c - \tfrac{3}{4} s^2 \rho_c \tag{5.8}$$

with

$$u_k = [1 + 54s^4 + 6(-1)^k s^2 (3 + 81s^4)^{1/2}]^{1/3}, \qquad k = 1, 2$$

In the limit $e \to 0$ one finds

$$\rho_c \sim \frac{1}{2e}, \qquad l_c^2 \sim \frac{1}{8e}$$

which are the Newtonian results for the classical Kepler motion. This will be discussed in more detail in the next section for arbitrary mass ratios.

Non-circular Motion

Now let $0 < l < l_c$, or, for fixed e, $l := \lambda l_c$ with $0 < \lambda < 1$ (then $\epsilon := (1-\lambda^2)^{1/2}$ is the eccentricity of the resulting ellipse in the Newtonian limit). It is easy to establish rigorously what is expected on physical grounds, namely that for decreasing λ the smaller positive root ρ_1 of $F = 0$ decreases (down to zero for $\lambda \to 0$), while the greater positive root ρ_2 increases. The mean radius $a := \frac{1}{2}(\rho_1 + \rho_2)$, or the 'major half axis', that is independent of λ in the Newtonian

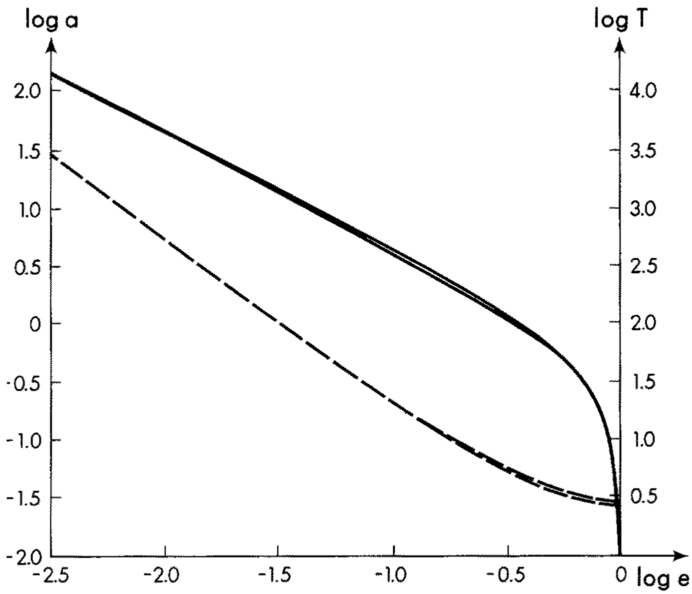

Figure 1.—The mean radius a and the half period T (dashed line) as a function of the binding energy e. The lower curve for both a and T corresponds to $\lambda = 0{\cdot}025$, the upper one to $\lambda = 0{\cdot}999999$. The straight line portions for small energy represent Kepler's third law.

case turns out to depend only slightly on λ even for extreme relativistic situations, while its dependence on e, of course, differs from the Newtonian one as e tends to 1. The mean radius a, as well as the 'half period'

$$T := \int_{\rho_1}^{\rho_2} \frac{G}{\sqrt{I}}\, d\rho$$

where obtained numerically and are plotted in Fig. 1.

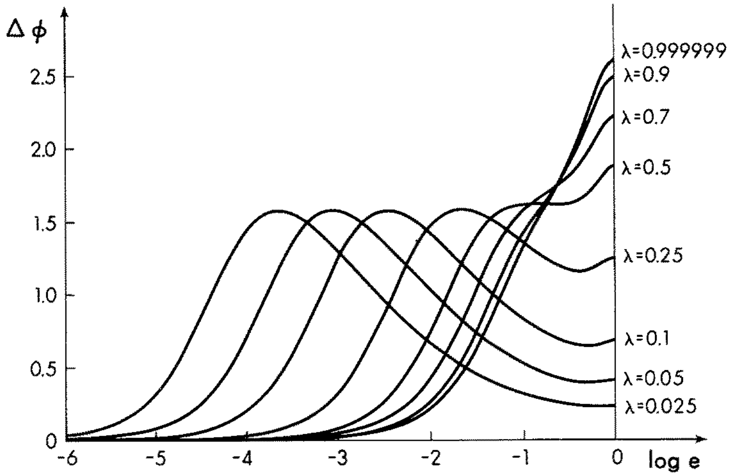

Figure 2.—The perihelion advance $\Delta\phi$ as a function of the energy for different angular momenta.

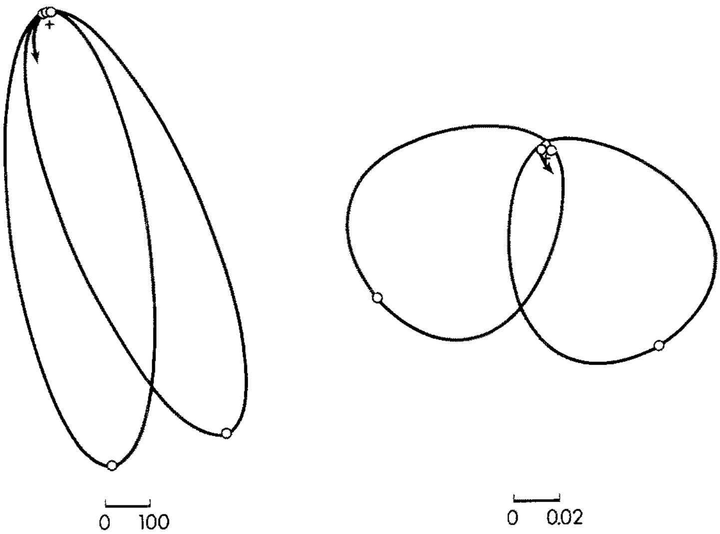

Figure 3.—Relative orbits for $\lambda = 0{\cdot}3$ and $e = 0{\cdot}001$ (left) and $e = 0{\cdot}95$. In the Newtonian limit λ is the quotient (minor half axis)/(major half axis) of the elliptic orbit. The units are the dimensionless ones of $\boldsymbol{\rho}$.

The results so far indicate that the most interesting quantity of the relativistic relative orbits is the perihelion advance $\Delta\phi = 2(\phi_1 - \pi)$ where

$$\phi_1 = \int_{\rho_1}^{\rho_2} \frac{H}{\sqrt{I}} d\rho$$

The perihelion advance as obtained by numerical integration is plotted as a function of e and λ in Fig. 2. For small λ the used integration procedure becomes more and more expensive for a given accuracy, but the trend is obvious from Fig. 2: As e tends to zero for fixed λ the perihelion advance also tends to zero, the Newtonian limit. If, however, $\lambda = 2l_0\sqrt{(2e)}$ as $e \to 0$, then $l \to l_0$. This is not the Newtonian limit, but instead the relativistic analogue of the Newtonian parabolic motion ($e = 0, l \neq 0$) and so there is no reason why the perihelion advance should vanish.

Orbits in the Center of Mass Frame

For specific values of e and λ equations (5.3) and (5.4) were integrated to obtain the functions $\phi(\rho)$ and $\tau(\rho)$ and thus the orbits of the relative motion. They look like rotating ellipses except for extremely relativistic binding energies when the orbits become more pear shaped (cf. Fig. 3).

Combining the results for the relative motion with equations (4.26) and (4.27) gives the two particle trajectories in the center of mass frame (Figs. 4 to 6). In the limit $e \to 0$ the corresponding Newtonian motion is again recovered. But for highly relativistic situations a certain asymmetry between the two trajectories shows up, which is due to our choice of describing the particle

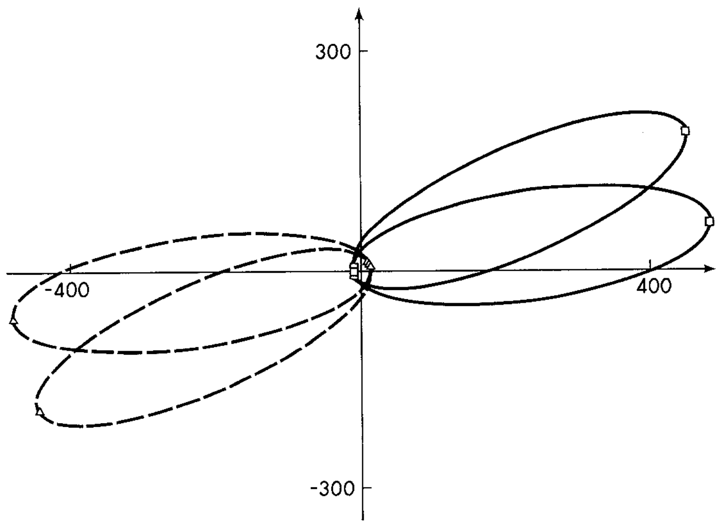

Figure 4.—Particle trajectories in the center of mass frame for $e = 0{\cdot}001$ and $\lambda = 0{\cdot}3$. The units are the dimensionless ones of y_1 and y_2. Two full cycles (perihelion to perihelion) are plotted. All perihelion and aphelion positions of the particles are indicated. The dashed line is the trajectory of particle 2.

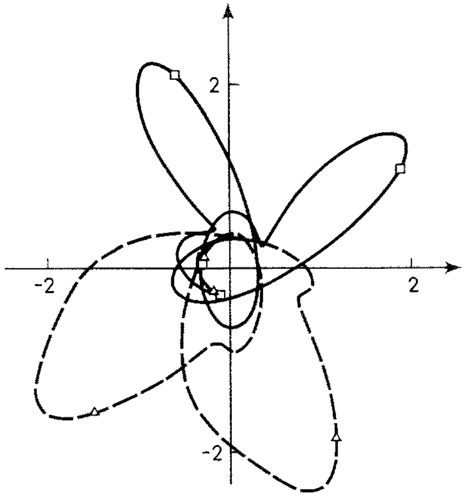

Figure 5.—Particle trajectories in the center of mass frame for $e = 0{\cdot}2$ and $\lambda = 0{\cdot}3$.

motions in advanced-retarded form. Note that for the ultra relativistic case ($e \to 1$) the two particles move jointly around the 'center' of mass, their relative distance being always much less than the distance from the center of mass.

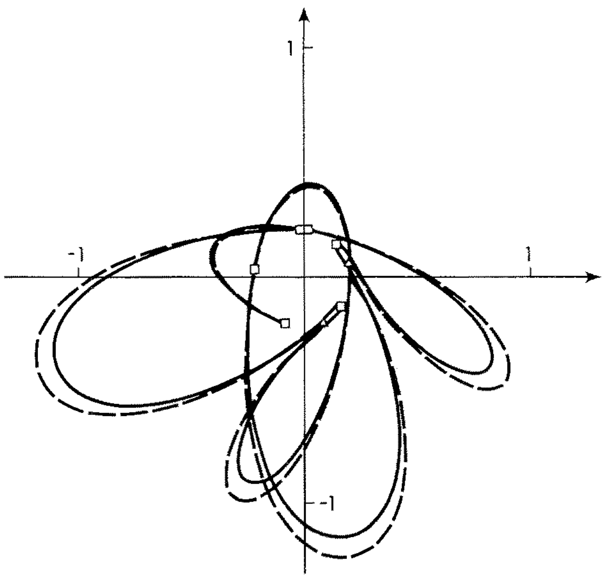

Figure 6.—Particle trajectories in the center of mass frame for $e = 0{\cdot}95$ and $\lambda = 0{\cdot}3$.

6. *Circular Motion for Arbitrary Mass Ratios*

The procedure indicated at the end of Section 4 for the solution of the equations of motion becomes extremely cumbersome when $\nu \neq 0$, because then Φ is a fourth-order polynomial in σ. We thus confine ourselves here to finding the conditions for circular motion in the case of an attractive force, bounded motion ($M < m$) and non-zero angular momentum. Some aspects of the circular motions of these systems have previously been studied by Bruhns (1973).

Consider the function $\Phi(\rho, \sigma, l)$ (keeping ν and e fixed all the time) as defining a family of curves in the ρ-σ plane, parametrized by l. Circular motion will result if such a curve is either parallel to the σ-axis or consists of an isolated point only. The first possibility can be excluded by a closer inspection of the function Φ. Since Φ is a polynomial in ρ, σ and l if follows that an isolated point (ρ_c, σ_c) must be a relative extremum of Φ, i.e. a solution of the system

$$\Phi(\rho, \sigma, l) = 0, \qquad \partial_\rho \Phi(\rho, \sigma, l) = 0, \qquad \partial_\sigma \Phi(\rho, \sigma, l) = 0 \tag{6.1}$$

Thus triples (ρ_c, σ_c, l_c) that solve these three polynomial equations and are such that $\rho_c > 0$, $|\sigma_c| < \pi_c = c\rho_c - \nu\sigma_c + 2$ and $l_c^2 > 0$ then yield the possible circular motions of the system. Note that then ρ_1 and ρ_2 are constant as well as the angular velocity

$$\omega_c = \dot{\phi}(\rho_c) = 2cl\alpha^2 \rho^{-1} \beta^{-1} \delta^{-1} |_{\rho_c, \sigma_c, l_c} \tag{6.2}$$

In practice the solutions of (6.1) were found by Newton's method starting with $\nu = 0$, where according to the last section $\sigma_c = 0$, and then changing all variables

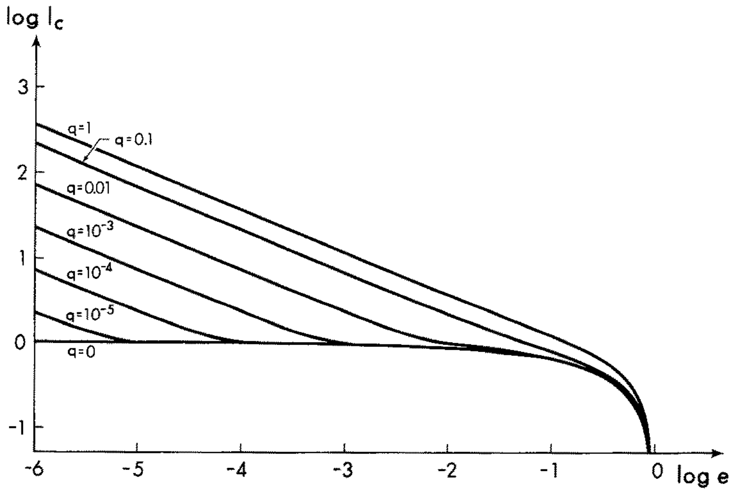

Figure 7.—The (dimensionless) angular momentum l_c for circular motion as a function of the energy.

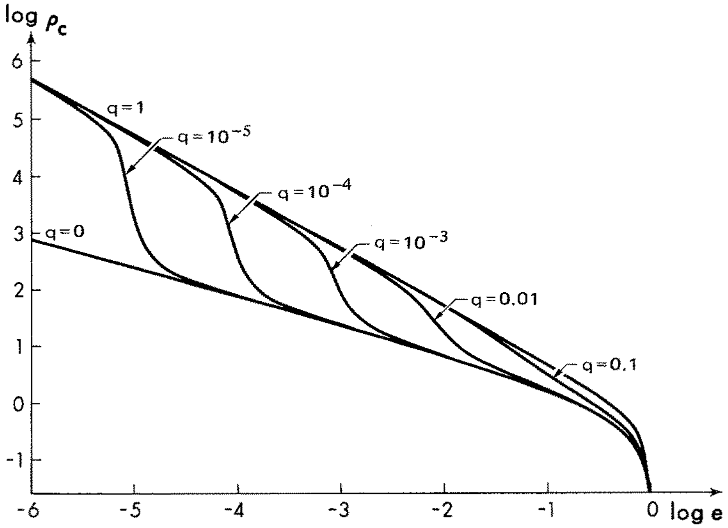

Figure 8.—The radius ρ_c of the circular (relative) orbit as a function of the energy.

gradually in order to reach only the physically significant solutions. In Figs. 7 and 8 the values of l_c and ρ_c are plotted as functions of the energy e for different values of the rest mass ratio $q = m_2/m_1$.

We indicate here how the asymptotic limits were obtained and show that Kepler's laws are indeed satisfied in the limit $e \to 0$, except in the case of infinite mass ratios. For fixed ν let, motivated by the case $\nu = 0$ (where $\sigma = 0$),

$$\rho_c = a_{-1} e^{-1} + a_0 + 0(e), \qquad \sigma_c = b_0 + 0(c)$$

and

$$l_c^2 = c_{-1} e + c_0 + 0(e)$$

Then equations (6.1) yield to the lowest order in e

$$\rho_c = \frac{1}{2e} [1 + 0(e)] \tag{6.3}$$

$$\sigma_c = -(\nu/\mu)[1 + 0(e)] \tag{6.4}$$

$$l_c = (\mu/8e)^{1/2} [1 + 0(e)] \tag{6.5}$$

If the definitions (4.2) to (4.11) of the dimensionless quantities are recalled it is easily verified that the energy dependence of the radius and the intrinsic angular momentum of the circular orbit for $E \ll m_1 + m_2$ are exactly those of the non-relativistic Coulomb field. Substitution of (6.1) into (6.2) gives to lowest order in e

$$\omega_c = 4e(2e/\mu)^{1/2} \tag{6.6}$$

or

$$\tau_c^2 := 4\pi^2/\omega_c^2 = \pi^2\mu^2/(8e^3) = \pi^2\mu^2\rho_c^3 \tag{6.7}$$

which is precisely the form Kepler's third law takes in dimensionless units.

We now consider the limit $q = m_2/m_1 \to 0$ or $\nu \to -1$. This limit should correspond to the situation that is normally referred to as the relativistic or electromagnetic Kepler problem (cf., for example, Synge (1965)). Unfortunately one cannot just put $\nu = -1$ in the basic equations of Sections 3 and 4. But motivated by the numerical solutions of equations (6.1) we expand for e fixed in powers of $\mu = 1 - \nu^2$,

$$\rho_c = \rho_0 + \rho_1\mu + 0(\mu^2) \tag{6.8}$$

$$\sigma_c = \sigma_{-1}\mu^{-1} + \sigma_0 + 0(\mu) \tag{6.9}$$

$$l_c^2 = l_0^2 + l_1^2\mu + 0(\mu^2) \tag{6.10}$$

Then equations (6.1) can be solved consistently and we find to lowest order in μ

$$\rho_c = (1-s)s^{-1}c^{-1} + 0(\mu) \tag{6.11}$$

$$\sigma_c = 2\mu^{-1} + 0(1) \tag{6.12}$$

$$l_c = (1-s)c^{-1} + 0(\mu) \tag{6.13}$$

whence also

$$\omega_c = sc^2(1-s)^{-1} + 0(\mu)$$

These equations were used to obtain the curves for $q = 0$ in Figs. 7 and 8. It is rather obvious from these figures that the limits $e \to 0$ and $q \to 0$ do not commute. In particular, the Newtonian limit of the infinite mass ratio relativistic Kepler problem cannot be taken by letting the binding energy go to zero.

Finally we note that an ansatz like (6.8) to (6.10) can be substituted directly into equations (4.32) to (4.34). New quantities $\bar{\rho} = \rho_0$ and $\bar{\sigma} = \sigma_{-1}$ can be introduced for the discussion of the general non-circular motions in the infinite mass ratio limit.

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